Cocenter of Hecke algebras of crystallographic Coxeter groups

Felix Schremmer

Joint work with: Xuhua He

The University of Hong Kong

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Hecke algebras

Take a crystallographic Coxeter group

$$W = \langle S \mid (st)^{m_{s,t}} = 1 \ \forall s, t \in S \rangle, \ \ m_{s,t} \in \{0, 1, 2, 3, 4, 6\}.$$

Example: Weyl groups, affine Weyl groups.

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Its Iwahori-Hecke algebra $H/\mathbb{Z}[\mathbf{q}^{\pm 1}]$ has basis $\{T_w \mid w \in W\}$. Multiplication determined by

- $T_v T_w = T_{vw}$ if $v, w \in W$ satisfy $\ell(vw) = \ell(v) + \ell(w)$,
- $T_s^2 = (\mathbf{q} - 1) T_s + \mathbf{q} T_1$ for $s \in S$.

Motivation: Finite type

Suppose W is finite. Let $G/\mathbb{F}_q$ be split group with Weyl group W, choose $G \supseteq B \supseteq T/\mathbb{F}_q$. Then

$$H_{\mathbb{C}} = \left\{ f\colon G(\mathbb{F}_q) \to \mathbb{C} \mid \begin{array}{c} \forall g \in G(\mathbb{F}_q) \\ \forall b_1, b_2 \in B(\mathbb{F}_q) \end{array} \ : \ f(b_1 g b_2) = f(g) \right\}$$

is the $\mathbf{q} = q$ specialization of our abstract Hecke algebra. Multiplication rule is

$$(f_1 f_2)(g) = \sum_{h \in G(\mathbb{F}_q)/B(\mathbb{F}_q)} f_1(h) f_2(h^{-1} g) \ , \ f_1, f_2 \in H_{\mathbb{C}}.$$

Motivation: Finite type, continued

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Tits deformation thm: $H_{\mathbb{C}}$ is semisimple and $\cong \mathbb{C}[W]$.

Tilting theory gives equivalence of categories

$$H_{\mathbb{C}}\text{-mod} \xrightarrow{\cong} \{V \in \mathbb{C}[G(\mathbb{F}_q)]\text{-mod} \mid V = \mathbb{C}[G(\mathbb{F}_q)]V^{B(\mathbb{F}_q)}\}.$$

Motivation: Finite type, continued

$$H_{\mathbb{C}} = \left\{ f\colon G(\mathbb{F}_q) \to \mathbb{C} \;\middle|\; \begin{matrix} \forall g \in G(\mathbb{F}_q) \\ \forall b_1, b_2 \in B(\mathbb{F}_q) \end{matrix} \;:\; f(b_1 g b_2) = f(g) \right\}$$

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$\rightsquigarrow$ The cocenter $\overline{H}_{\mathbb{C}} = H_{\mathbb{C}}/\mathrm{span}_{\mathbb{C}}\{h_1 h_2 - h_2 h_1 \mid h_1, h_2 \in H_{\mathbb{C}}\}$ gets a perfect pairing

$$\overline{H}_{\mathbb{C}} \otimes \mathbb{C}[V \in \mathrm{Irr}(G(\mathbb{F}_q)) \mid V^{B(\mathbb{F}_q)} \neq 0] \to \mathbb{C}.$$

Cocenter in general

We return to the abstract Hecke algebra $H/\mathbb{Z}[\mathbf{q}^{\pm 1}]$ for (W, S) any crystallographic Coxeter group.

Theorem (Geck-Pfeiffer, He-Nie, Marquis).

(a) *Let $\mathcal{O} \subseteq W$ be a conjugacy class. Then for any two minimal length elements $w_1, w_2 \in \mathcal{O}$, we have $T_{w_1} = T_{w_2}$ in $\overline{H}$.*
$\rightsquigarrow$ *obtain $T_{\mathcal{O}} \in \overline{H}$.*

(b) *The set $\{T_{\mathcal{O}} \in \overline{H} \mid \mathcal{O} \in \mathrm{Cl}(W)\}$ spans $\overline{H}$ as $\mathbb{Z}[\mathbf{q}^{\pm 1}]$-module.*

(c) *If W is finite or affine, then then $\overline{H}$ is a free $\mathbb{Z}[\mathbf{q}^{\pm 1}]$-module with basis as in (b).*

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Theorem (He-S., in preparation).

Part (c) is true for any crystallographic Coxeter group (W, S).

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There exists a "group" G with max. torus T, "Borel" B and Weyl group W.
Then $G(\mathbb{F}_q)$ is profinite with $B(\mathbb{F}_q)$ open compact. Pick Haar measure $\mu(B(\mathbb{F}_q)) = 1$. Let

$$H_{\mathbb{C}} := \left\{ f\colon G(\mathbb{F}_q) \to \mathbb{C} \text{ c.s.} \mid \begin{array}{c} \forall g \in G(\mathbb{F}_q) \\ \forall b_1, b_2 \in B(\mathbb{F}_q) \end{array} f(b_1 g b_2) = f(g) \right\}.$$

This is the $\mathbf{q} = q$-specialization of H and $\overline{H}_{\mathbb{C}} = \overline{H}_{\mathbf{q}=q}$.

Corollary.

$\overline{H}_{\mathbb{C}}$ has $\mathbb{C}$-basis $\{ T_{\mathcal{O}} \mid \mathcal{O} \in \mathrm{Cl}(W) \}$.

Trace functions

Goal: For each $\mathcal{O} \in \mathrm{Cl}(W)$, construct a function $f : \overline{H} \to \mathbb{Z}[\mathbf{q}^{\pm 1}]$ with

$$f(T_{\mathcal{O}'}) = \delta_{\mathcal{O},\mathcal{O}'},\ \forall \mathcal{O}' \in \mathrm{Cl}(W).$$

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If W is finite, consider

$$\begin{aligned} \mathrm{Tr}(h_1, h_2) &:= \mathrm{tr}(H \to H,\ h \mapsto h_1 h h_2) \\ &= \sum_{w \in W} \text{coeff. of } T_w \text{ in } h_1 T_w h_2 \end{aligned}$$

for $h_1, h_2 \in H$.

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for $h_1, h_2 \in H$. This yields a map

$$\mathrm{Tr} : \overline{H} \otimes \overline{H} \to \mathbb{Z}[\mathbf{q}^{\pm 1}]$$

and $\{\mathrm{Tr}(T_{\mathcal{O}}, -) \mid \mathcal{O} \in \mathrm{Cl}(W)\}$ are linearly independent.

Parabolic induction

Let $I \subseteq S$, generating $W_I \leq W$ with Hecke algebra $H_I/\mathbb{Z}[\mathbf{q}^{\pm 1}]$.
Given a function $f\colon \overline{H}_I \to \mathbb{Z}[\mathbf{q}^{\pm 1}]$, consider

$$H \to \mathbb{Z}[\mathbf{q}^{\pm 1}], h \mapsto \sum_{\substack{w_1 \in W^I \\ w_2 \in W_I}} f(T_{w_2}) \text{ coeff. of } T_{w_1 w_2} \text{ in } hT_{w_1}.$$

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For certain pairs (I, f), this gives a well-defined function $\overline{H} \to \mathbb{Z}[\mathbf{q}^{\pm 1}]$.

Goal: For all $\mathcal{O} \in \mathrm{Cl}(W)$ find $f\colon \overline{H} \to \mathbb{Z}[\mathbf{q}^{\pm 1}]$ s.th. $f(T_{\mathcal{O}'}) = \delta_{\mathcal{O},\mathcal{O}'}$

$\mathcal{O}$ elliptic, $\mathcal{O} \cap W_I = \emptyset \; \forall I \subsetneq S$
Can write down expl. traces again using results of Krammer & Marquis

$\mathcal{O}$ has finite order

Orbital Integrals

Consider our maximal Kac-Moody group G.
Idea: For $s \in G(\mathbb{F}_q)$ of finite order prime to q ("semisimple"), consider

$$H_{\mathbb{C}} \to \mathbb{C}, \ f \mapsto \int_{\{g^{-1}sg | g \in G(\mathbb{F}_q)\}} f \, d\mu.$$

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(In practice, we integrate over a slight thickening of the conjugacy class of s).
Question: Can such maps distinguish all finite order conjugacy classes?
$\rightsquigarrow$ Suffices to show this for W finite.

A problem on finite groups of Lie type

Let W be finite, $G/\mathbb{Z}$ corresponding Chevalley group. Is it true for infinitely many q that semisimple conj. classes of $G(\mathbb{F}_q)$ separate the cocenter $\overline{H}_{\mathbb{C}}$? I.e. do the functions

$$\overline{H}_{\mathbb{C}} \to \mathbb{C}, \ f \mapsto \sum_{g \in G(\mathbb{F}_q)} f(g^{-1}sg), \ s \in G(\mathbb{F}_q)^{\mathrm{ss}}$$

span $\mathrm{Hom}_{\mathbb{C}}(\overline{H}_{\mathbb{C}}, \mathbb{C})$?

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span $\mathrm{Hom}_{\mathbb{C}}(\overline{H}_{\mathbb{C}}, \mathbb{C})$?
Answer: Impossible if G has a Levi subgroup of type E_7.
Otherwise yes for all $q \gg 0$, and we only need regular semisimple elements.

Summary of proof

Goal: $\forall \mathcal{O} \in \mathrm{Cl}(W)$, find $f\colon \overline{H} \to \mathbb{Z}[\mathbf{q}^{\pm 1}]$ s.th. $f(T_{\mathcal{O}'}) = \delta_{\mathcal{O},\mathcal{O}'}$.

1. Parabolic induction $\rightsquigarrow$ only need elliptic & finite order
2. Elliptic classes can be handled directly
3. $\mathcal{O}$ finite order $\rightsquigarrow$ pick minimal $I \subseteq S$ with $\mathcal{O} \cap W_I \neq \emptyset$.

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    - I does not contain $E_7 \implies$ Orbital integrals do the job
    - I contains $E_8 \implies$ parabolic induction still works
    - I contains E_7 but not $E_8 \implies$ need truncated version of parabolic induction AND orbital integrals

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Thank you for listening!

Elliptic Classes

Let $\mathcal{O} \in \mathrm{Cl}(W)$ be elliptic.
Goal: Find $f : \overline{H} \to \mathbb{Z}[\mathbf{q}^{\pm 1}]$ with $f(T_{\mathcal{O}'}) = \delta_{\mathcal{O},\mathcal{O}'}$.
Marquis: For any $w \in \mathcal{O}$, some power $b = w^n$ will be *straight*.
$\rightsquigarrow$ build "semi-infinite" H-representation M^b
Krammer: If (W, S) is irreducible and non-affine and $w \in \mathcal{O}$, then $\langle w \rangle \leq C_W(w)$ has finite index.
$\rightsquigarrow$ find well-defined trace functions on M^b, essentially

$$h \mapsto \sum_{z \in W/\langle b \rangle} \lim_{n \to \infty} \text{coeff. of } T_{zb^n w} \text{ in } hT_{zb^n}$$

Semisimple classes in $G(\mathbb{F}_q)$

Let $s \in G(\mathbb{F}_q)^{\mathrm{rss}}$

- Write the map $G(\mathbb{F}_q) \to \mathbb{C},\ g \mapsto \begin{cases} 1, & g \in \{s\}, \\ 0, & g \notin \{s\} \end{cases}$ as $\mathbb{C}$-linear combination of Deligne-Lusztig characters.
- Summing a character $\chi : G(\mathbb{F}_q) \to \mathbb{C}$ over Schubert cells $B(\mathbb{F}_q)wB(\mathbb{F}_q)$
  $\cong$ restrict to a character $H_{\mathbb{C}} \to \mathbb{C}$
  $\cong$ project χ onto the space of principal series unipotent char.
- $\implies \#(B(\mathbb{F}_q)wB(\mathbb{F}_q)) \cap \{s\} = \text{const. } R^1_{C_G(s)}(T_w)$.
- So rss. conjugacy classes separate $\overline{H}_{\mathbb{C}} \iff$
  - Every rat. max. torus in G has regular & rational element and
  - the square matrix

$$(\text{principal series unipotent char.} | \text{unipotent DL char.})$$

is invertible.